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AUTHOR McKnight, Curtis C.
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ABSTRACT

The focus of this paper is on the use of frames in achieving understanding of learning and problem solving in school mathematics. The paper seeks to illustrate the use that has been made of frames and frame-like conceptions and to add to the case for the usefulness of such conceptions. Results reported are: (1) those related to more sequential procedural schemata; (2) those related to some less sequential schemata; and (3) evidence of schemata-like interpretation of algebraic expressions in the early stages of processing. (MP)

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The Use of "Frames" in Conceptualizing Learning and Problem Solving in Mathematics

Curtis C. McKnight

Curriculum Laboratory

University of Illinois at Urbana-Champaign

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This paper is closely related to the paper "Conceptualizing the Structures Underlying Cognitive Behavior - The Usefulness of 'Frames'" by Davis, with both being first presented in the same symposium. Davis focuses on the usefulness of "frames," which he uses to refer to the common general conception behind those using "frames," "scripts," or "schemata" in recent work in cognitive science (Minsky, Schank, Rumelhart and Ortony, etc.). The present paper, by contrast, focuses on the use of "frames" ("scripts," "schemata," etc.). Both papers arise out of work done for the last several years at the Curriculum Laboratory at the University of Illinois at Urbana-Champaign. A report of the first phase of that work is available in Davis, Jockusch, and McKnight (1978).

Strategy of the Research

In that work, a clear first priority was given to the documentation of a wide range of phenomena occurring in mathematics classes dealing with algebra and geometry. This first step is considered vital in achieving understandings of mathematics learning that have clear relevance and practical import for actual classrooms. However, it was also recognized that this was only a first step, and that in order to move from phenomena documentation to understanding of learning and problem solving in (school) mathematics, one must go on to consideration of appropriate explanatory mechanisms.

The strategy adopted in the present work is the "white box" strategy of proposing explanatory models rather that the more traditional "black box" (input-output) approach of earlier learning theories. That "white box" approach is common to the emerging cognitive science (as represented by the authors referenced earlier) and is explained and defended more fully by Davis (1979). For the present stage of our work "frames" (and the others) must be considered as (at least initially) posited metaphoric models of children's information

processing which, for now, do the following: (1) provide strong heuristic guidance in analyzing documented phenomena, and in seeking significant interrelationships among aspects of those phenomena; and (2) provide an appropriate rhetoric for that analysis.

This strategy perhaps faces the danger of reification but this danger is minimized by utilizing the above strategy as only one component in a more complex strategy, the three phases of which may be characterized as follows:

(1) Documentation of a wide range of classroom and individual phenomena in student learning and problem solving in algebra and geometry (the documentary approach representing relatively a-theoretical phenomena-capturing by naturalistic observation [participant and non-participant] and by clinical interviews and collection of talk-aloud protocols.);

(2) Phenomena analysis and posing of explanatory models which have an, at least, a priori case for being useful, i.e., heuristically potent and rhetorically rich; and

(3) More precise exploration of a more narrowly targeted set of phenomena for which [1] alternative models are examined and [2] data is gathered related to those points most critical in deciding between alternative models.

Thus far in the work reported here, a great deal of the first phase has taken place (although more is going on) and a fair amount of the second. The third phase remains for the next few years in this work.

The remainder of this study reports a sampling of results from the second phase in the above strategy. It is, in a sense, an interim report. It seeks to (1) illustrate the use that has been made of frames and frame-like conceptions; and (2) add to the case for the usefulness of such conceptions. If at times the results do not seem definitive (or, perhaps even unambiguous) their status as interim products of the above three-phase strategy must be remembered.

It is, of course, possible to adopt an alternative strategy of beginning in the first phase with a narrowly targeted set of phenomena (e.g., those related to arithmetic or algebraic word-problems) which then allows the next phase to be a conflation of our second and third phases. Such a strategy would appear in some ways more penetrating and efficient and should probably be exploited in some cases, especially those most concerned with exploring the application of some specific theoretical mechanism. Our strategy, which is at least initially more diffuse, is motivated, first, by our relatively agnostic position on how well a wide range of mathematics learning has been understood and, second, by our primary concern with effecting significant changes in school mathematics learning and instruction (rather than a primary concern more appropriate to others, that of furthering theory [which is at best a secondary concern for us].)

The remainder of the paper, then, reports some "interim" results as follows: (1) results related to more sequential "procedural" schemata; (2) results related to some less sequential schemata; and (3) evidence of schemata-like interpretation of algebraic expressions in the early stages of processing.

Procedural Schemata

Within the general domain of our use of schemata one important sub-domain is that of using them to explore the phenomena of sequential processes as discussed in Davis (1979). Schemata, as they are usually conceptualized, are networks of related sub-schemata or "variable slots" which may be "bound" to (considered as being exemplified by and mapped onto) features in a given situation to which the schemata are being applied. As a network, there is no implied sequence of movement through the schema and that fact helps give schemata their utility in explaining a variety of cognitive processes. However, there is nothing in the conception that limits it to non-sequential behaviors and it is often fruitful to think of an individual's algorithmic problem-solving behavior as based on a schema with the network restricted or structured to produce sequential behavior (i.e., performance based on the "script" represented by the schema will exhibit sequential regularities). Such procedural schemata open up many possibilities in explaining algorithmic-like human behavior. Two illustrative aspects of our studies of such phenomena are discussed here.

A. Procedure Clarification. Davis (1979) writes about the execution of "procedures" as one kind of sequential process that can be explicated by schemata, considering a procedure as the human analog of the program of something like a programmable calculator. Related to this, he distinguishes between sub-procedures (analogous to computer sub-routines) and super-procedures (analogous to main programs which guide overall processing and the calling of sub-routines in their proper place).

We have explored procedures, especially the relation of super-procedure to sub-procedures, in looking at the learning of some of the more algorithmic aspects of school algebra. One technique in those explorations might be called the "before and after snapshot." Brief (about five minutes) quasi-clinical interviewing was used to generate talk-aloud protocols for students solving two

or three routine problems and to gather student clarifications of ambiguous points in the protocol. "Before" and "after" pictures were obtained by conducting two interviews on the same type of problem, the first within the first day or so of instruction relevant to the problem type, and the second towards the end of immediately relevant instruction. One such "before and after snapshot" is discussed here, but we have collected many such protocols, of which this one if fairly typical.

The subject was a seventh-grade boy, whom we shall call "P", who had been in an introductory algebra course for about two and a half months. The course had covered signed numbers, some beginning analytic geometry, logic and formal derivations of algebraic expressions, and it was just then moving into instruction on simplifying algebraic expressions prior to going into the arithmetic of polynomials. Thus, tasks like

"Simplify $4x^2 + 2x - [x^2 + 3x]$"
or "Simplify $5x^2 + 3x - [x^2 + 4x]$"

represented new, challenging problems, although some preparation had been made during the course of instruction on algebraic derivations (which involved use of formal axioms like the distributive law, commutative law of addition, etc.).

Let us first look at P's "after" picture, which came on the eighth day of instruction on the topic, just before moving on to the next topic. Given the task, "Simplify $5x^2 + 3x - [x^2 + 4x]$," his protocol may be summarized as follows:

(1) He first announced that he was going "to make the minus into plus a negative" and wrote

$$5x^2 + 3x + {}^-[x^2 + 4x].$$

[Notice, he unhesitatingly calls up and executes this simple syntactic transformation procedure. He possessed a verbal label for what he did, had no doubts that it was a legal syntactical move, and knew that it would be helpful to him overall, suggesting that he has a clear super-procedure embedding this step.]

(2) Next he annouces that he is going to handle the "parenthesis" and writes

$$5x^2 + 3x + {}^-x^2 + {}^-4x.$$

[Again, clearly a legal syntactic transformation based on the distributive law, which P executes without question or hesitation, indicating the functioning of an organized super-procedure. Instruction had suggested this transformation as one of several that would be legal and effective in what, in meta-language, would be called "removing parentheses." Stress was placed on a meta-language discussion of the super-procedure

rather than extensive drill on executing one form of it. Notice P calls the brackets here "parentheses," even though the instructor and the interviewer had called them "brackets." The meta-language discussion has "taken," at least in part.]

(3) P next uses the commutative law of addition twice to bring his "like" terms together, writing

$$5x^2 + -x^2 + 3x + -4x.$$

[Again, he does this unhesitatingly and without pause between the two executions, indicating a clear super-procedure. He mentions that he can now do this, since it's "addition," which suggests he has some awareness of why the first two steps were necessary. The effect of formal derivations on P's procedure is interesting. Further work typically curtails a step like this one, leaving it implicit in the step which follows, "combining like terms." Thus, this step indicates that the structure of P's super-procedure has been shaped by his experience in formal algebraic derivatives in which he has made moves like this numerous times. This is strange in comparison to (1) the fact that, asked to name the justifying law here, he comes up finally with the associative law rather than the commutative, and (2) the confusions in this "before" protocol, which will be discussed below.]

(4) Finally, he twice adds "like" terms, writing the final result,

$$4x^2 + -1x.$$

[Again, clear super-procedure and correct sub-procedures. Interestingly, the final "-1x" in his answer reveals the carefulness and precision of his syntactical transformations. Such carefulness is also curtailed with further experience and confidence. Again, such clarity about syntactical matters will be seen to be in marked contrast to his "before" protocol.]

This "after" protocol seems to be obvious and to be a simple exercise in rhetoric application. The usefulness of the approach becomes clear in contrasting the "before" protocol to this precise, efficient, correct target performance in the "after."

The "before" protocol was taken at the beginning of the third (45 minute) period of instruction on the topic. P had worked simpler exercises and had seen worked exercises something like his assigned task, "Simplify $4x^2 + 2x - [x^2 + 3x]$." His "before" protocol may be summarized as follows:

(1) P announces that he wants to do something with the "like terms," identifying the $4x^2$ and x^2 as such, but he is concerned that "it's minus that [x^2] so...?" Later he expresses confusion about the fact that the 4 and the x^2 in the $4x^2$ are combined by multiplication, while the 3x and the x^2 in $x^2 + 3x$ are combined by addition. [Notice, P is at first unable to call any sub-procedure; his "so..." trails off into a question. He has no super-procedure ready at hand. At best, he seems to have some heuristic guidance in the form of knowing he needs to "combine like terms." This leads him to reflect on the expression and his reflecting is a strange mixture of syntactic awareness and confusion. He realizes the minus should tell him something, but he's

not sure what. He contrasts the connectives in $4x^2$ and $x^2 + 3x$, which is astute in the sense that most similar students would not have that clear a grasp of the syntax of $4x^2$ or be led to compare the two sub-expressions. Yet the contrast is almost esoteric in its irrelevance to the needed procedure and shows a confusingly wide-ranging, undirected reflection.]

(2) After noting "you can't add," he copies down $4x^2 + 2x$ without combining unlike terms as he briefly considered, and then goes on to complete the expression,

$$4x^2 + 2x - 3x + x^2.$$

[P's explicit heuristic of combining "like terms" (apparently with a further implicit heuristic of bringing "like terms" into proximity, as we see correctly done in a full-blown form in step 3 of the "after") and his syntactical confusions lead him to take the plunge with an incorrect, ad hoc procedure. He has the syntax of $4x^2 + 2x$ correct, and even the commutation of $x^2 + 3x$ would be correct inside parentheses. He seems to have no awareness of the syntactical function of the parentheses, nor any legitimate procedure for removing them. Thus, the ad hoc move that is guided only by rough heuristics, not an organized super-procedure, and which shows lack of knowledge of a key syntactical point or else a lack of use of such knowledge in a validity check of the simple ad hoc procedure he has concocted (to relieve the pressure on him?).]

(3) P next said that he had to "take" the 3x from both the 2x and the $4x^2$. He then wrote a -1 below the 2x and indicated that $2x - 3x$ gave -1. When questioned further, he stated that "you minus the x [in 3x] from 2x." [Having followed his rough heuristics to the point where they can now dictate directly a step, P proceeds to follow up his syntactical blunder in the preceding step with a further syntactical blunder. This is a clear case of some sort of regression under strain, because in previous days P had easily done related simpler problems correctly, using the same procedure which would be the appropriate sub-procedure here. Thus, encountering sub-problems which he should be able to do in a context that is already demanding much of his cognitive resources causes malfunctions in his syntactical understandings and in his sub-procedures.]

(4) P now moves confidently to take 3x from $4x^2$, writing 1x below the $4x^2$. Then he suddenly expresses some doubt that this is correct and, after a pause, decides it is not, because "$x^2 - x$" doesn't "equal x." He then goes back to consider whether he needs to take the 3x from both terms and eventually decides he does not.

[The confidence with which he begins this step seems to show that P has at least a rudimentary super-procedure to cover a few steps. The lack of hesitation seems a clue to such super-procedures. Suddenly his rough heuristic about "like terms" seems to enter in a different way, acting as a validity check which leads him to reject what he has just done. It is not clear why he is able to go on to realize that he was making a previous error in even taking 3x from $4x^2$. A reasonable hypothesis, based on previous evidence of the good level of knowledge P has about algebraic syntax, is that it is this syntactical knowledge that surfaces here. We have often seen that one error will start children on a chain of errors like some short-term, localized algebraic dementia. P's earlier wrong transformation seems to have done this, but encounter with some syntactical fact of which he is sure brings him out of it and leads him to correct another of his errors.]

(5) P then rewrites $4x^2 + 2x + 3x + x^2$ (to give)
$4x^2 - x^2 + 2x - 3x$.
[Somehow, "like terms" has put him back on the right track and he can access and use the same sub-procedure that he uses in step 3 of the "after" and, if the previous steps had been correct, it would be appropriate here. Unlike step 3 of "after," this sub-procedure is not executed as a part of a super-procedure but rather as a part of something like a "visually moderated sequence" (Davis, 1979) in which visual and other outputs from a previous step lead to selection and execution of a sub-procedure even without an overall plan.]

(6) Finally, P moves quickly to write 4 as the output of $4x^2 - x^2$ and $^-1$ as the output of $2x - 3x$, giving him a final answer of $4 + ^-1$, which does not trouble him at all. [This, of course, is the application twice of the same inappropriate procedure seen in step 3 of this protocol. It is mysterious why P can snap out of pressure- or error-induced regression enough to correct two errors but not go on to correct a third. It certainly goes to show the persistence of some errors. One possible difference is that while P has performed a sort of validity check both in step 3 and in step 5, that of step 5, "$x^2 - x$" is correct and shows his error, while that of 3, "you must minus the x from the 2x" is a syntax-ignoring, misleading verbalization. Such a verbalization may have short-circuited the validity check process and left the error-producing sub-procedure intact.]

While P's final answer is, of course, atrocious, and while he showed several shortcomings in this "before" performance, his actual errors are few and easily corrected. It simply took pointing out the error in steps 3 and 6 to make P realize that it was wrong and, even to him, silly [although other students have proved more resistant to such knowledge], and a brief bit of instruction about the syntax of parentheses and about their removal. At least one practical benefit of the super/sub-procedure analysis is to pinpoint errors sufficiently precisely to greatly aid in remediation.

Comparison of P's "before" and "after" protocols suggests several conclusions (which are supported by other related protocols), including the following:

(1) The key task in tasks like that of P is really that of <u>super-procedure formation.</u> Probably the strongest contrast between before and after is that of the confidence and lack of hesitation in the after, which is justified by the correct sequence of sub-procedure accessings.

(2) Execution of already formed procedures as sub-procedures in such a context (i.e., super-procedure formation) often results in a decay in performance of the already-formed procedures. This is similar to what Bruner and others have found in motor skills learning. It may make a case for over-learning of procedures that will be sub-procedures in a variety of more complex procedures.

(3) The first two phenomena combine in a change which might be called mutual clarification, i.e., when a correct super-procedure is stably formed, so will be correct sub-procedures. While it was not wholly the case with P, in some cases we have seen group instruction which continued to focus on forming the correct super-procedure result in seemingly spontaneous correction of sub-procedures.

(4) Formation of super-procedures seems clearly to run from trial-and-error crystallization around a few heuristic guides to a stable, stereotypic structure. This might be called an exodigmatic (paradigm from "within") formation approach, as contrasted to an hypodigmatic (paradigm from "above") approach (i.e., repeated imitation of an overall model).

(5) The syntax of algebraic expressions may be a key or milestone kind of knowledge in algebraic learning. The degree of syntactic security seems to be a crucial element in a student's predisposition to regression under strain.

(6) Formal derivations (such as these students had been exposed to) may help in creating secure syntactic awareness and transformations, even though it seems often to fail at the task of making the student able to provide explicit justifications.

B. Cue Structure Errors. Within the context of all schema-based cognitive processing, but especially in that related to sequential behavior, a key source of student errors has been that of cue utilization. Procedures and sub-procedures are accessed and their execution is guided by utilizing as cues various features of the task representation and interim products (especially in "visually moderated sequences"; Davis, 1979). This is discussed in more detail in Davis, Jockusch, and McKnight (1978). Here we will present only the sketch of one protocol that illustrates the fruitfulness of the line of inquiry opened by this aspect of schema-based processing.

A common technique in our work has been the use of lengthier (30-40 minute), more fully clinical interviews to survey the work of students with difficulties in order to identify and document common error types. Such interviews are frankly much more exploratory than those mentioned earlier. The interview to be discussed here involved a seventh-grade girl, whom we will call "A", and the transcript involves 340 utterances and runs to 19 typed pages. We will summarize only the highlights here. A more complete discussion of the transcript is available from the author.

A's class had been working on polynomial arithmetic for some time and A was having considerable difficulty, especially with long multiplication of polynomials (the class had moved on to long division of polynomials). To begin the interview A was given the task of multiplying $2x^4 - x^2 + x - 3$ and $x^3 - x + 2$. After some initial difficulties, A wrote the problem as follows:

$$\begin{array}{r} 2x^4 - x^2 + x - 3 \\ \underline{\qquad x^3 - x + 2} \end{array}$$

[Notice this already suggests that there may be some structural confusions for A, since it is more typical to write

$$\begin{array}{r} 2x^4 \quad - x^2 + x - 3 \\ \underline{x^3 \qquad - x + 2} \end{array}$$

.]

A then went on to multiply like this:

$$\begin{array}{r} 2x^4 - x^2 + x - 3 \\ x^3 - x + 2 \\ \hline 4x^4 \ -2x^2 \ +2x - 6 \end{array}$$

At this point she expressed some confusion as to whether 2 times $2x^4$ was really $4x^4$, stating that it seemed that she had only multiplied coefficients.

A then begins the next line saying, "So you naturally move it over one..." and writing:

$$\begin{array}{r} 2x^4 - x^2 + x - 3 \\ x^3 - x + 2 \\ \hline 4x^4 \quad -2x^2 \ +2x - 6 \\ \qquad\quad -3x \end{array}$$

The type of small sign-assignment error found here is typical of A and many students, and it certainly makes it difficult to pinpoint even more crucial errors. Some time was spent correcting A's mistake, and she then went on, writing:

$$\begin{array}{r} 2x^4 - x^2 + 3 - 3 \\ x^3 - x + 2 \\ \hline 4x^4 \quad -2x^2 \ +2x - 6 \\ +x^3 - x^2 \ +3x \end{array}$$

At this point, in writing the "$+x^3$", A became very uncertain about where it should be located (although she finally chose correctly). This led the interviewer to decide to explore her knowledge of the structure underlying the columns. This generated the following exchange:

I: What name would you give for this column here? [pointing to the first column on the right]

A: That's the first column.

I: What about this column? [the x column]

A: The second column.

I: Can you give it [the second column] a name that seems to take more account of what you're putting in the column? What are you putting in this column?

A: Oh...uh...x?

I: And this first column here? [the right column again]

A: Negative three?

A's obvious confusion and seeming lack of knowledge about column structure then lead to a brief episode in which the interviewer tries direct instruction. This, too, quickly ends in noncomprehension on A's part. This leads the interviewer to try to help A discover the commonalities in a column, which produces the following exchange:

I: How is this column here with $-2x^2$ and $-x^2$ different from the 2x and 3x column?

A: Well, it's a negative but...

I: True. Tell me another way it's different.

A: Well, it's got an exponent.

I: What is the exponent?

A: Two.

Certainly A demonstrates some pattern recognition ability, but she also shows no real understanding of which feature of column structure commonality is essential at this point and, in fact, she shows a willingness to use multiple criteria.

This led the interviewer to turn aside to a relevant classification task in an attempt to pin down more precisely which features A used as cues and what organizations this led her to. He wrote the following terms: $-3x$, $2x^2$, x^3, x, $-7x$, x^2, $-x^2$, and $3x^4$. He then asked A to "put those in columns" [and further exchanges make clear that A understood this in terms of columns for a long multiplication task]. After several exchanges, A completes the task and produces the following array of possible columns:

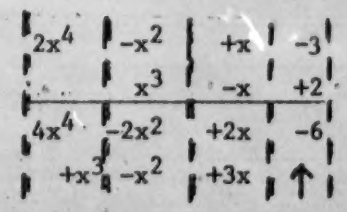
x	$-3x$	x^3	$-2x^2$
	$-7x$	x^2	$3x^4$
		$-x^2$	

At first this array appears to be an inconsistent, ad hoc complex rather than any logical classification. However, the following analysis of A's classification indicates strongly that this is likely not the case:

	Exponent Absent	Exponent Present
Coefficient Present	$-3x$, $-7x$	$-2x^2$, $3x^4$
Coefficient Absent	x	x^3, x^2, $-x^2$

A has, in fact, produced a perfectly logical multiple classification using two features of which she is aware and ignoring a third (presence or absence of minus). The features she uses have been significant in other contexts (as has been the one she does not use), so A's behavior is not simply a matter of using "perceptual" or "surface" features, and her behavior is very likely not an immature "heaping" into "complexes."

A's classification behavior makes it clear that she is certainly not dominated by the correct understanding of column structure for the algorithm and which cue that structure makes salient. The next question then becomes whether A's lack of understanding was promoted only by the absence of appropriate knowledge or by such absence plus something A regards as legitimate information which is, in fact, wrong and contradictory to the relevant structural information. In actuality, later analysis of the protocol reveals that we have already been given an essential clue that the latter is the case. If we turn back to the last answer A gives in the first dialogue presented earlier ("negative three?") and compare it to the visual array with which she was working, something interesting comes to light. Consider the following:



The vertical dashed lines were, of course, not in A's array. The arrow indicates the position of the interviewer's finger in indicating the column of interest that led to A's "negative three" response. The interviewer (and an observer) initially applied that mathematically "mature" understanding that one only considers the column structure of the answer, i.e., below the horizontal line, and they assumed A was doing the same. It came as a marked shock on reanalysis of the protocol to discover that A's answer indicated that she was extending the columns into the problem statement as well, i.e., that she was considering a column as being all of the entries between a pair of dashed lines (which we have put in for clarity). In that case, given the way that A had originally written the problem with x^3 under $-x^2$ and where she had written the $4x^4$ and $+x^3$ in the answer, she has correctly perceived that no single criterion will explain the within-column commonalities for all of the columns. This makes the consideration of multiple criteria a logical next step.

Of course, it is true that the double classification A used in the classification task would not explain correctly the column structure she perceived (the $-2x^2$ and the $+x^3$ would be out of place). What we are claiming is that once one is led to think of A's behavior in terms of procedures, features, and cues, it immediately begins to appear more rational and the errors A makes are more precisely pinpointed. She did not comprehend the appropriate basis for within-column commonality and she produced a misleading problem representation (misleading, not incorrect). This led her to rightly reject any single feature commonality criterion. Given a related classification task, she then takes the next step of applying consistently a double classification that utilizes the two features most salient in her perception of the problem array, one which does, in fact, cover most of the cases in that array. This seems to represent both a tremendous drive towards rationality on A's

part and an effective use of some powerful pattern recognition and classification tools. Such an analysis seems clearly to show not that children (or, at least, A, although she is quite typical) are blatently illogical or unintelligent but, rather, that their errors are often very localized and small in comparison to the powerful rational processes they utilize.

The localized (but often subtle and/or persistent) nature of children's errors is shown by the rest of the episodes with A. After the classification task, the interviewer began an interlude of remediation in which he attempted to lead A to discover the correct <u>basis</u> for within-column commonality and thus the correct cue/feature for classification. A quickly realizes that what we want to be able to do is put "things we can add together" into the same column. A reveals some further difficulties in applying that criterion, e.g., adding x and -3x to produce $-3x^2$. A demonstration of the distributive law eventually debugs that faulty procedure and, with her addition working right, A quickly comes to see that the exponent is the salient cue in this case. She is confident of this and, when asked, promptly returns to the earlier classification task and re-sorts the terms correctly relative to the exponent criterion. Nowhere in this remediation was an explicit correction of A's extension of columns into the problem statement (since the interviewer did not recognize it until later), but apparently the interviewer's continued use of only answer terms and his evocation of the correct basis were sufficient to eliminate this (probably temporary) procedure.

Yet even at this late stage, A is still hiding a subtle confusion. She returns to the original multiplication task and continues it as follows:

$$\begin{array}{rrrr} 2x^4 & -x^2 & +x & -3 \\ & x^3 & -x & +2 \\ \hline 4x^4 & -2x^2 & -2x & -6 \\ -2x^5 + x^3 & -x^2 & +3x & \\ & -3x^3 & & \end{array}$$

At this point disaster has seemingly struck again. Her error leads to the following exchange:

I: How did we decide a minute ago? You told me what made the distributive law work in some cases and not work in other cases.

A: Well, if they've got an exponent then it'll work.

I: Well, not just an exponent..How did you decide here? [her second classification answer] Which are the things that you put in the same column with a $-2x^2$? Are the things squared?

A: Oh, so this is wrong because [it's not squared].

I: Right...and now let's put it in the right place.

A: O.K. It would go right here.

I: You're right.

A: Oh, I see how this works! I've never understood this before!

Clearly the previous work had led A to correct her criterion explicitly to only something like "exponent or not," although she used the correct, more precise criterion in her second classification. Finally, her last error and the exchange it prompted leads her to see completely the structure that she had been working with. The excitement of her last remark makes it clear that this is some gestalt-like breakthrough for her (and her later work bore this out). If anything, this last difficulty seems to make it clear that (1) not only are children's errors often localized, but also their corrections, which makes it necessary to expend further effort to get them to cross-reference the correction to other parts of the relevant schemata, and (2) implicit use of debugged procedures (as in A's second classification) is often not enough to guarantee lasting behavioral change and, in fact, effort must be spent either to overlearn the debugged procedure or raise the correction to an explicit awareness so that change is at the long-term cognitive structure level rather than a short-term behavioral level with such behavior based on some temporary, "work space" cognitive structure.

Schemata Formation and Use

Our work has been concerned not merely with schemata that may be the basis of sequential, especially algorithm-like, behavior. We have also been concerned about schemata consisting of more complex networks which embed not simply variables that can be related to simple perceptual features (like exponents in a polynomial) but, rather, variables that are themselves schemata, often representing something more conceptual and complex (and containing several sub-variables of their own). The formation and use of such more complex schemata likely provide efficient explanations of children's encounters with novel, non-routine problems.

Collecting phenomena that are likely to be related to such schemata formation and use has proven to be a difficult problem, not because the phenomena are scarce (in fact, we think we could easily find enlightening "live" examples of almost any aspect of this process), but, rather, because the documentation of such phenomena is so difficult. The economics of our research has thus far made it necessary to work with audiotaped rather than videotaped interviews. Unfortunately (for documentation) many of the key features of such phenomena are non-verbal and quite subtle. Also, perhaps because the context has a great deal to do with the overall strategy chosen by children, it has often proven difficult to induce in a documenting interview setting some of the creative, individualized problem-solving phenomena we have seen occurring in classrooms and in out-of-class, informal settings.

Even those phenomena we have documented require lengthy transcripts and discussions, which presents an additional problem in trying to share them in brief as must be done here. For now, we must be content with "telling" rather than "showing," and attempt to communicate the flavor of our explorations in this area. Please bear in mind that our documentation is more complete than we will be able to share and that, also, refinements toward greater precision are continually going on in this stage of our work.

To begin the "telling," we will report on problem solving by a 13 year old girl (call her K) and a 13 year old boy (call him T). In one study, both were interviewed on solving the following two problems:

(1) a "falling body" problem of the type "If an object is thrown straight up with an initial velocity of 50 fps, how high will it go?"

(2) A variant "falling body" problem asking one, given the maximum height, to find the initial velocity.

The first of these two problems was part of the current classwork for these two very good problem solvers.

K, on being given the first problem and asked to "talk through it" without working it, gave an initial "a ha" reaction (which we take to mean some kind of schema accessing; it is quite common) and immediately, without hesitations, went through a meta-language description of the solution of the problem using variables in the place of numbers and writing very little. T performed almost exactly the same in his interview. This seems to be an example of problem solving that involves accessing some appropriate, somewhat generalized schema and applying it. Our request for a meta-language presentation was an attempt to get at schema-accessing as opposed to application (since a numerical application would be hard to distinguish from some piece-wise accessed "visually moderated sequence").

The second problem was novel to K and T but related to familiar problems. The same underlying schema of the scenario would apply to give the same set of general descriptive equations, but the variables that must be tagged as "to find" would be changed. This would necessitate accessing slightly different equation-solving sub-procedures at the end.

K started into her meta-discussion quickly, but at a few key places paused thoughtfully. She was, however, with those few pauses able to carry out correctly a meta-discussion describing how she would modify her approach to the

first problem in order to solve the second. T, on the other hand, began doubtfully and soon stopped in confusion. He then proceeded to execute a numerical solution of the problem (again with a few key longer pauses) that embodied the classical write-pause-write pattern that we take as evidence of a visually moderated sequence.

We consider K to have begun with the underlying schema or schemata from her first problem and to have gone about the business of surveying the variables of those structures. She paused precisely in the places where changes in variable bindings and taggings were needed and at the place where she had to begin to describe the slightly different equation solving. She was successful, it seems to us, in accessing a relevant schema and modifying it to apply directly to the problem at hand. T, on the other hand, was unable to carry out the modification of the schema at the meta-discussion level. He needed a more concrete "work space" and shifted to a visually moderated sequence. This allowed him to more routinely use a variable-binding routine to generate his equations and then use the interim products (the equations) to cue familiar equation solving routines. T effectively transformed the problem from a generalized, conceptual problem involving some high-powered schema utilization procedures, turning it into a standard, fairly algorithmic algebraic problem for which he could conflate existing procedures in a piece-wise fashion to come up with an adequate procedure. More detailed discussions of this study are presented in Davis, Jockusch and McKnight (1978).

An interesting related phenomenon actually occurred at the end of K's interview. She asked that the recorder be shut off and then asked that the interviewer (who also was her teacher) go over with her again a type of problem concerning components of vectors from the week's classwork. She led the teacher through the problem, then back over selected parts of it, making several

remarks of a "So, in general, you would do such-and-such" form. She then sat quietly in thought for several minutes. This kind of rehearsal and re-playing of problem-solving performances, seemingly to "schematize" them and produce new cognitive structures, is common not only for K but for many students. That long (several minute) silent perusal of a solved problem has been documented repeatedly, usually by students who then prove easily able to routinely solve such problems in the future. One boy, D, regularly turns in a note the morning after a test (from which he would have possessed no written records), making a variety of comments, both conceptual and numerical, about things he did wrong or did not at first understand clearly (but which he now explains well) on the test. He apparently is able to re-play some sort of episodic record of his performance (as expert chess players often are able to do), to reflect on it and analyze it, and, presumably, to abstract from it some sort of more semantic memory representations useful in guiding future problem solving.

Schema-based Interpretive Process

Probably the previous section's discussion seemed almost impressionistic (although we believe the full documentation to be less so). Perhaps that discussion can be balanced by one of the more precise types of our studies which will be presented here.

Frame and schemata theories generally conjecture that interpretation of complex perceptual inputs commonly takes place by relating them to some appropriate frame or schema, probably by some combination of top-down and bottom-up information processing. If that is actually the case in children's mathematical behavior, what might be some reasonable evidence that such was the case?

Schema theories typically assume that variables in the schemata will have default values which will be assigned to the variable if nothing in the

(perceptual) input is bound to the variable. If perceptual interpretation situations can be manipulated in such a fashion that schema accessing can begin but variable binding not be completed, this should make default assignments occur. If this can be done in a context in which the likely default values are known (but differ from what is actually in the perceptual input), evidence could be gathered of such default assignments (in that case, default errors) which can be taken to imply at least some element of top-down, schema-based processing.

Further, it is a reasonable hypothesis, in many contexts, that complex, appropriate cognitive structures (i.e., schemata) will be possessed and utilized by subjects experienced in the area, while those with less experience will be less likely to have formed relevant structures and, therefore, more likely to engage in data-driven, bottom-up processing than in conceptually-driven, top-down processing. In some contexts such schema-based processing, more typical for experienced subjects, may result in errors that would not have occurred if a more primitive, piece-meal processing was used. Such errors might be called "errors of sophistication" since, presumably, the more sophisticated or experienced subject is more likely to make such errors. Default errors presumably would be just one type of "error of sophistication."

We have been able to document several "errors of sophistication" in our work. Sometimes the explanation of such errors still remains highly conjectural. However we have also been able to document carefully a number of default errors, the explanation of which in schemata terms is considerably less conjectural. Several of those findings are presented here.

In one study, a test containing 14 signed number questions was given to 56 public junior high school (eighth and ninth grade) algebra students. A subset of nine of the items dealt only with addition and subtraction of signed numbers. After the tests were scored, a rank ordering of the nine items by

difficulty was made. A separate rank ordering was made for those students scoring 5 or more (i.e., above 50%) on the test (32 students). Differences in the two rank orderings which show items more difficult for the high-scoring group would seem to be likely sources of errors of sophistication, especially if the increased difficulty comes from one common error made by the high-scoring group and not the rest.

Three rank order differences were found, out of which two indicated possible errors of sophistication (item eight was third most difficult for the high group, fourth for the whole group; item three was 6th for high group, 8th for whole group). The following is the breakdown of common errors for the two items:

Item #	Item	Whole Group Answer %		High Group Answer %	
		Response	%	Response	%
8.	-1-5=	-6*	48	-6*	63
		(+)4	21	(+)4	28
		(+)6	9	(+)6	9
		-4	4		
		3	2		
		None	16		
3.	-5+4	-1*	80	-1*	84
		-9	11	-9	16
		(+)1	3		
		4.5	2		
		None	4		

*Correct answer

Notice that for item 8, the difficulty difference is entirely due to the fact that 28% of the high group vs. 21% of the whole group gave the wrong answer, (+)4. Similarly, in item 3, it is due to the fact that 16% vs 11% gave the wrong answer, -9. This localization of the difficulty difference to one more common wrong answer in each case makes it likely that these are errors

of sophistication. What procedure is most likely to produce the two wrong answers? One reasonable conjecture (which has been seen "in motion" in some interviews) is the following:

$$-(1-5) = +4$$
$$-(5+4) = -9$$

If, in fact, a larger number of the higher scoring students are seeing an "implied" parenthesis in each item, both of the more common wrong answers are explained. Notice that this is a precise syntactical move, since in item eight they are careful to preserve the order, i.e., 1-5 gives -4, not +4 as would be gotten if the more naive procedure was used for looking only at a (symmetric) difference between the two numbers. This result is at least strongly suggestive that these higher scoring students are getting into trouble purely because of their superior (but inadequate) knowledge of a syntactical feature, that they are inappropriately applying that structure in a situation of uncertainty. This finding is evocative but perhaps not convincing.

Another study involved a simple procedure to try to generate default errors of sophistication in interpreting algebraic expressions. The format involved the investigator being seated on one side of a 3 foot wide table across from a single student. Algebraic expressions were written in letters about 1.5 cm high on white 4 x 5 cards. These were manually placed and removed from the center of the table with a digital timer being used to provide the subject with 2 seconds exposure to the expression followed by a 10 second writing period before the next expression was exposed. The following instructions were read to each subject:

> "Please fill in your name and the date. I will be showing you a series of algebraic expressions for about 2 seconds each. I will turn each card face up for 2 seconds and then face down. Like this.[Demonstration with a sample card containing 2 (3+5) -5] As I show you each card, I want you to

> copy as closely as possible the exact expression that you have seen. I will give you 10 seconds between each expression so that you will have time to write down all that you can. There will be 16 expressions. Please write them in order on your answer sheet. [Order indicated on form for answers] Do you have any questions?

Typically there were no questions, except three students who asked the question, "Exactly what I've seen?" In almost all cases the 10-second writing period was sufficient, with most students having 1-2 seconds to prepare for the next item. Thus, omissions were almost always due to memory rather than time considerations. Generally, students watched during the 2 seconds, beginning to write only after the exposure ended.

It was hoped that this fairly standardized, tachistoscopic-like controlled exposure paradigm would allow the accessing of available interpretive schemata, but would not allow complete variable binding. Only a few items were included for which a priori there seemed a reasonable likelihood that interpretive schemata would be accessed to reduce memory strain. The other items served primarily a secondary purpose of trying to get some rough measure of feature-by-feature channel capacity for algebraic expressions.

Sophistication was manipulated in two ways in the experiment. A group of 12-13 year old ninth grade algebra students and a comparison group of 16-17 year old seniors in a calculus class were tested. Further, about one half of the items were expressions of algebraic factors like $(x+2)(x+3)$, while the other half involved algebraic fractions like $(x+2)/(x+3)$. The ninth grade group was tested after a factoring unit, before an algebraic fractions unit, and then a second time after the fractions unit to test for any recency effects. In fact, the factoring items were mostly too simple to produce errors for most subjects, so for the senior group and the ninth grade "after" group, an additional four, more complex items were added, making a 20-item test. Tables 1 through 5 at the end of the paper present some results.

A simple sort of default error is the sign error in reproducing an expression like the items shown in Table 1. All subjects consistently recognized the polynomial forms involved sufficiently that they typically assigned a sign even when they could not recall the correct one. All but two or three of the sign omissions came when a subject curtailed reproducing the expression, i.e., before curtailment sign assignments were made rather than leaving blanks. To test explicitly such defaults, item 19 deliberately contains a missing sign in the denominator (see Table 1). Only 12 of 49 subjects tested got that far in reproducing the expression, but of those 12, only 1 actually left a blank appropriately, and she thought it so unusual that she asked the experimenter afterwards if he had intended to do that.

Such a low level default error says little about use of any stored symbol structure schemata, since the form is present in the stimuli (in all cases except item 19's denominator). A slightly more complex default error occurred in item 5 which was $\frac{x^2 - 9}{3 - x}$.

Of the 49 subjects, four wrote x-3 rather than 3-x. This suggests that 3-x is coded something like "the monomial x-3, reversed." That is in line with a recall study we conducted, one aspect of which asked a wide sampling of ninth through twelfth graders to recall and write the quadratic formula. The most common error by far (occurring in almost 40% of the cases) was to write $4ac - b^2$ rather than $b^2 - 4ac$. This again suggests storage of expressions with a possible "reversed or not" tag. In very short-term memory and more so in long-term memory, the content of the tag can decay, with the subject then assigning the most likely default value (as in item 5) or perhaps assigning one of two possible default values randomly (as in the quadratic formula).

Perhaps the most telling default error is the fact that in well over 100 exponent errors, all except 2 were reductions of exponents to produce something closer to the more familiar quadratic or monomial forms. Thus, responses to the numerator of item 9 (see Table 1) were often $2x^2 + 3x^2 + 5x$, $2x^3 + 3x + 5x$, $2x^3 + 3x^2 + 5$, and, especially, $2x^2 + 3x + 5$. This overwhelming predominance of reductions rather than random exponent errors suggests that most represent default assignment of exponents in accord with the values to which students had been most often exposed, namely, quadratic and monomial forms.

Incidentally, tables 2 through 5 can tell us something about whether such errors are errors of sophistication or not, and about how students utilize their processing resources. Table 2 suggests there is little difference due to increasing experience; with a few possible exceptions as follows: (1) more experienced subjects tend to make fewer omissions (guess more often and/or make more default assignments); and, (2) there do seem to be recency effects, since in several cases the ninth grade "after" percentages are better than those for the twelfth grade. [It is, of course, possible that the reason that clear error of sophistication effects are not seen is because an insufficient range of sophistication was included. The ninth grade students are actually beginning their second year of an algebra course and have worked with polynomial expressions the previous year. Perhaps the experiment should be tried again, using truly algebraically inexperienced students.]

If one compares (in Table 1) item 17 to the numerator of item 9 and the denominator of item 18, one finds a cubic expression in three different positions which might well have an effect on reproduction error percentages. Presumably the cubic alone (item 17) and the cubic in the numerator (item 9) should produce close to the same error pattern with a possible decrement for

the numerator. The cubic in the denominator (item 18), typically processed last, would likely show a greater performance decrement (i.e., rising error and omission percentages). Table 3 presents the error percentages for cubic expressions in the three positions, while Table 4 presents corresponding omission percentages. The omissions do follow the pattern that was predicted, but the errors do not. In fact, the error percentages follow no discernable pattern. This suggests that either our measurements are not yet sufficiently sensitive or else that children's use of processing resources are not as rational or patterned as one might have expected.

To cloud the picture further, Table 5 presents the results for item 19, which has quadratic expressions of roughly equal difficulty in numerator and denominator. Again, there is no marked performance decrement in the denominator except for the twelfth grade sign errors, which is explained by their greater proclivity for assigning a default value to the blank (8 of the 11 errors of this type were in this group). The twelfth graders show a tendency to make fewer omissions, which may explain their higher error percentages compared to the ninth graders in the denominator errors. This is true also in Tables 3 and 4, where the denominator columns show the older subjects (e.g., 13% omissions, 43% errors for signs) doing the same thing compared to the younger (e.g., 47% omissions, 19% errors). This tendency may explain some of the lack of consistency in the various tables. Potential error percentage patterns seem to be confounded with omission percentage patterns.

Concluding Remarks

The preceding examples certainly make clear the interim nature of many of our results to date, but they should also serve to make clear some of the uses to which frame-like conceptions may be put in explaining children's learning and problem solving in mathematics. Clearly they provide an analytical

rhetoric. They also appear fruitful in precisely characterizing and diagnosing children's errors, and seem to make more powerful, effective remediation available. The conceptions seem to fit well with the phenomena we have observed and to provide at least heuristic guidance in characterizing problem-solving behavior and individual differences in it. Further, while no definitive studies have been done yet, frame conceptions certainly seem fruitful in suggesting more precise tests to gather evidence of the veridicality of such conceptions in describing children's mathematical behavior.

Finally, even this survey, and certainly our fuller explorations of the uses of frame-like conceptions, suggest strongly their usefulness. Frame-like conceptualizations seem clearly to offer a powerful theoretical, metaphoric rhetoric for developing models of mathematics learning, problem solving, and comprehension, having great practicality, e.g., in evaluation and diagnosis of individual students. At the very least, our findings convince us sufficiently to lead us to continue to pursue the three phase strategy described earlier.

Item #	Item
9	$\frac{2x^3 + 3x^2 + 5x}{x^3 - 8x}$
17	$4x^3 + 3x^2 - 7x$
18	$\frac{x^2 - 4}{x^3 + 5x^2 + 6x}$
19	$\frac{3x^2 - 2x + 5}{x^2 + 7x \quad 4}$

Table 1. Items 9, 17, 18 and 19

		Errors			Omissions		
		9 Before	9 After	12	9 Before	9 After	12
Numerator	Coefficients	9	2	2	17	6	7
	Exponents	33	13	22	17	2	2
	Signs	11	9	13	19	19	3
Denominator	Coefficients	3	3	10	75	72	67
	Exponents	14	16	17	75	72	67
	Signs	11	0	20	78	81	67

Table 2. Results for Item 9. (Percentages of possible cases)

	9 After			12		
	Alone (17)	Num. (9)	Den. (18)	Alone (17)	Num. (9)	Den. (18)
Coefficients	10	2	17	7	2	13
Exponents	8	13	13	4	22	13
Signs	41	9	19	30	13	43

Table 3. Errors in $Ax^3 \pm Bx^2 \pm Cx$ in Items 9, 17 and 18. (Percentages of possible cases)

	9 After			12		
	Alone (17)	Num. (9)	Den. (18)	Alone (17)	Num. (9)	Den. (18)
Coefficents	2	6	50	0	7	42
Exponents	2	2	50	0	2	42
Signs	0	19	47	0	3	13

Table 4. Omissions in $Ax^3 \pm Bx^2 \pm Cx$ in Items 9, 17, and 18 (Percentage of possible cases)

		Errors		Omissions	
		9 After	12	9 After	12
Numerator	Coefficients	4	4	4	0
	Exponents	0	0	4	0
	Signs	13	13	0	0
Denominator	Coefficients	2	9	77	44
	Exponents	2	2	77	47
	Signs	9	27	78	50

Table 5. Results for Item 19 (Percentages of possible cases)

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